Problem 3

(a) Show that $\tan \frac{1}{2}x = \cot \frac{1}{2}x - 2\cot x$.

(b) Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n}$$

Solution

Part (a)

$$\tan \frac{1}{2}x = \cot \frac{1}{2}x - 2\cot x$$

The aim is to change all the terms to sine and cosine and then to show that the two sides are equal. The half-angle formulas for tangent and cotangent are the following.

$$\tan \frac{1}{2}x = \frac{1-\cos x}{\sin x}$$
$$\cot \frac{1}{2}x = \frac{\sin x}{1-\cos x}$$

And cotangent is, by definition,

$$\cot x = \frac{\cos x}{\sin x}.$$

So for the right-hand side, we have

$$\begin{aligned}
\cot \frac{1}{2}x - 2\cot x &= \frac{\sin x}{1-\cos x} - 2\frac{\cos x}{\sin x} \\
&= \frac{\sin^2 x - 2\cos x(1-\cos x)}{\sin x(1-\cos x)} \\
&= \frac{\sin^2 x - 2\cos x + 2\cos^2 x}{\sin x(1-\cos x)} \\
&= \frac{1-\cos^2 x - 2\cos x + 2\cos^2 x}{\sin x(1-\cos x)} \\
&= \frac{1 - 2\cos x + \cos^2 x}{\sin x(1-\cos x)} \\
&= \frac{(1-\cos x)^2}{\sin x(1-\cos x)} \\
&= \frac{(1-\cos x)}{\sin x} \\
&= \tan \frac{1}{2}x.
\end{aligned}$$

Therefore,

$$\tan \frac{1}{2}x = \cot \frac{1}{2}x - 2\cot x.$$

Part (b)

If we write out the first few terms of the series, we see tangents whose arguments are consecutively halved. We'll use the formula we proved in part (a) to write them in terms of cotangent.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{2^n}\tan\frac{x}{2^n} &= \frac{1}{2}\tan\frac{x}{2}+\frac{1}{4}\tan\frac{x}{4}+\frac{1}{8}\tan\frac{x}{8}+\cdots\\ &= \frac{1}{2}\left(\cot\frac{1}{2}x-2\cot x\right)+\frac{1}{4}\left(\cot\frac{1}{4}x-2\cot\frac{x}{2}\right)+\frac{1}{8}\left(\cot\frac{x}{8}-2\cot\frac{x}{4}\right)+\cdots\\ &= \underbrace{\frac{1}{2}\cot\frac{x}{2}-\cot x}_{n=1}+\underbrace{\frac{1}{4}\cot\frac{x}{4}-\frac{1}{2}\cot\frac{x}{2}}_{n=2}+\underbrace{\frac{1}{8}\cot\frac{x}{8}-\frac{1}{4}\cot\frac{x}{4}}_{n=3}+\cdots\end{aligned}$$

As can be seen from the last step, the series is telescoping. Every n value gives us two terms, thanks to the trigonometric identity. The first term of one n value will always cancel the second term of the $n+1$ value. The terms up to $n=3$ are written out, and the only terms that don't cancel are $-\cot x$ and $(1/8)\cot(x/8)$. Thus, we can evaluate the series by calculating a limit.

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\tan\frac{x}{2^n}=-\cot x+\lim_{n\to\infty}\frac{1}{2^n}\cot\frac{x}{2^n}$$

To find the limit, write out the Taylor series of $\cot(x/2^n)$. As $n\to\infty$, only the first term in each series doesn't go to zero.

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{2^n}\cot\frac{x}{2^n} &= \lim_{n\to\infty}\frac{1}{2^n}\frac{\cos\frac{x}{2^n}}{\sin\frac{x}{2^n}}\\ &= \lim_{n\to\infty}\frac{1}{2^n}\frac{1-\frac{1}{2!}\left(\frac{x}{2^n}\right)^2+\frac{1}{4!}\left(\frac{x}{2^n}\right)^4-\cdots}{\left(\frac{x}{2^n}\right)-\frac{1}{3!}\left(\frac{x}{2^n}\right)^3+\frac{1}{5!}\left(\frac{x}{2^n}\right)^5-\cdots}\\ &= \lim_{n\to\infty}\frac{1-\frac{1}{2!}\frac{x^2}{2^{2n}}+\frac{1}{4!}\frac{x^4}{2^{4n}}-\cdots}{x-\frac{1}{3!}\frac{x^3}{2^{2n}}+\frac{1}{5!}\frac{x^5}{2^{4n}}-\cdots}\\ &= \frac{1}{x}\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\tan\frac{x}{2^n}=-\cot x+\frac{1}{x}.$$